

# Protometrics

In honor of Ivo Rosenberg

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## Abstract

We introduce the concept of protometric and present some properties of protometrics.

This note is a tribute to Ivo, a friend and co-author of the first author during last 40 years. It is written in the taste and style of Ivo, on the border of Logic, Distance Spaces and Combinatorics.

**Definition 1.** For a set  $X$ , we say that a function  $d: X \times X \rightarrow \mathbb{R}$  satisfies the *triangle inequality of type*:

o (outgoing)	iff	$d(x, y) + d(x, z) \geq d(y, z);$
i (incoming) <sup>1</sup>	iff	$d(y, x) + d(z, x) \geq d(y, z);$
t (transitive) <sup>2</sup>	iff	$d(y, x) + d(x, z) \geq d(y, z);$
c (cyclic)	iff	$d(z, x) + d(x, y) \geq d(y, z)$

for all  $x, y, z \in X$ . These will also be termed  $*$ -triangle inequalities, where  $*$  are type letters.

## Simple facts about the triangle inequalities

1. If  $d(\cdot, \cdot)$  is symmetric, i.e.,  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , then the four versions of the triangle inequality are equivalent.
2. Let  $d'(x, y) = d(y, x)$  for all  $x, y \in X$ . Then  $d'(\cdot, \cdot)$  satisfies the  $i$ -triangle inequality iff  $d(\cdot, \cdot)$  satisfies the  $o$ -triangle inequality (and vice versa). The  $t$ -triangle inequality holds or fails for  $d(\cdot, \cdot)$  and  $d'(\cdot, \cdot)$  simultaneously, and so does the  $c$ -triangle inequality.
3. If  $d(\cdot, \cdot)$  satisfies the triangle inequality of type:

$$\begin{aligned}
 o, \quad & \text{then} \quad d(x, x) \in \left[ \sup_{y \in X} (d(y, x) - d(x, y)), 2 \inf_{y \in X} d(y, x) \right]; \\
 i, \quad & \text{then} \quad d(x, x) \in \left[ \sup_{y \in X} (d(x, y) - d(y, x)), 2 \inf_{y \in X} d(x, y) \right]; \\
 t, \quad & \text{then} \quad d(x, x) \in \left[ 0, \inf_{y \in X} (d(x, y) + d(y, x)) \right]; \\
 c, \quad & \text{then} \quad d(x, x) \in \left[ \sup_{y \in X} |d(x, y) - d(y, x)|, \inf_{y \in X} (d(x, y) + d(y, x)) \right]
 \end{aligned}$$

for all  $x \in X$ . Hence<sup>3</sup>  $d(x, x) \geq 0$  for types  $t$  or  $c$  and  $d(x, y) \geq 0$  for types  $o$  or  $i$ .

4. If  $[d(x, y) = 0 \Leftrightarrow x = y]$  (identity of indiscernibles) and  $d(\cdot, \cdot)$  satisfies the triangle inequality of type  $o$ ,  $i$ , or  $c$ , then  $d(\cdot, \cdot)$  is symmetric and nonnegative, thus,  $d(\cdot, \cdot)$  is a metric.

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<sup>1</sup>It is also called the *strong triangle inequality*.

<sup>2</sup>It is also called the *oriented triangle inequality* or simply the *triangle inequality*.

<sup>3</sup>Moreover, for types  $t$  or  $c$ ,  $d(x, y) \geq 0$  holds if the symmetry of  $d(\cdot, \cdot)$  is additionally assumed.

**Definition 2.** For a set  $X$ , a function  $p: X \times X \rightarrow \mathbb{R}$  satisfies the *pre-quadrangle inequality* of type:

$$\begin{array}{ll} \text{o (outgoing)}^4 & \text{iff } p(x, y) + p(x, z) \geq p(y, z) + p(x, x); \\ \text{i (incoming)} & \text{iff } p(y, x) + p(z, x) \geq p(y, z) + p(x, x); \\ \text{t (transitive)}^5 & \text{iff } p(y, x) + p(x, z) \geq p(y, z) + p(x, x); \\ \text{c (cyclic)} & \text{iff } p(z, x) + p(x, y) \geq p(y, z) + p(x, x) \end{array}$$

for all  $x, y, z \in X$ . Such a function  $p(\cdot, \cdot)$  is called a *protometric* of the corresponding type. If the inequality is strict whenever  $z = y$  and  $y \neq x$  [2], then  $p(\cdot, \cdot)$  is a *strict protometric*.

### Simple facts about protometrics

1. The pre-quadrangle inequality of each type strengthens the triangle inequality of the same type if  $[p(x, x) \geq 0, \text{ but } p(x, x) \not\equiv 0]$  and reduces to it whenever  $p(x, x) \equiv 0$ . Any metric is a protometric of each type.
2. If  $p(\cdot, \cdot)$  is a protometric of type o, i, or c, then  $p(x, y) \geq \frac{1}{2}(p(x, x) + p(y, y))$  and  $p(\cdot, \cdot)$  is symmetric. If  $p(\cdot, \cdot)$  is a protometric of type t, then  $p(x, y) + p(y, x) \geq p(x, x) + p(y, y)$ . Thus, there are only two types of protometrics: general protometrics (of type t) and symmetric protometrics (of type o-i-t-c) also called, in the case of  $p(x, x) \geq 0$ , weak partial pseudo-metrics [6].
3. For any  $f: X \rightarrow \mathbb{R}$ ,  $p(x, y) = f(x)$  and  $p(x, y) = f(y)$  are protometrics. If  $p$  is a protometric, then so is  $p'(x, y) \stackrel{\text{def}}{=} p(y, x)$ . If  $p$  and  $q$  are protometrics on  $X$ , then so is  $p + q$ . Hence  $p(x, y) + p(y, x)$  is a symmetric protometric whenever  $p$  is a protometric.
4. Let  $p'(x, y) = \alpha p(x, y) + f(x) + f(y)$  for all  $x, y \in X$ , where  $\alpha > 0$  and  $f: X \rightarrow \mathbb{R}$ . Then  $p$  and  $p'$  are or are not protometrics of the same type simultaneously.
5. If in Fact 4 it holds that  $f(x) = -\frac{\alpha}{2}p(x, x)$ , then  $p$  is a protometric of any type iff  $p'$  satisfies the triangle inequality of the same type.
6. It follows from Facts 2, 3, and 5 that for any protometric  $p$ , the symmetric function  $d(x, y) = \alpha(p(x, y) + p(y, x) - p(x, x) - p(y, y))$ ,  $\alpha > 0$ , satisfies the triangle inequality, is non-negative, and  $d(x, x) \equiv 0$ . If, additionally,  $p$  is a strict protometric, then  $d(x, y) > 0$  whenever  $y \neq x$  and thus,  $d$  is a metric.

### Protometrics and similarity measures

The concept of protometric as applied to dissimilarity measures  $d$ , is somewhat exotic, since  $d(x, x) = 0$  is characteristic of such measures. However, the fulfillment of the pre-quadrangle inequality is quite typical of the function  $-s(\cdot, \cdot)$ , where  $s(\cdot, \cdot)$  is a similarity measure.

One example is the *Gromov product similarity* (or *covariance*)  $(x, y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$ , where  $d(\cdot, \cdot)$  is a metric and  $x_0 \in X$  is any fixed *base point*. It follows from the above Facts 1 and 4 that  $-(x, y)_{x_0}$  is a non-positive symmetric protometric (cf. the *Farris transform metric*  $C - (x, y)_{x_0}$ , where  $C$  is a large enough positive constant [5, Chapter 4]).

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<sup>4</sup>Appeared in [2] and some earlier papers by the same authors as the *triangle inequality for proximities*; cf. [1].

<sup>5</sup>This inequality was considered by Matthews [7]. It is also called the *sharp triangle inequality* and the *modified triangle inequation*.

Moreover, in [3] a number of proximity measures  $s(\cdot, \cdot)$  for graph vertices were presented such that  $-s(\cdot, \cdot)$  are non-positive protometrics.

A function  $s: X \times X \rightarrow \mathbb{R}$  satisfies the *transition inequality* if  $s(y, x)s(x, z) \leq s(y, z)s(x, x)$ . In [4], it was shown that a number of positive proximity measures  $s(\cdot, \cdot)$  for the vertices of a strongly connected weighted digraph satisfy the transition inequality. Consequently, the corresponding functions  $-\ln s(\cdot, \cdot)$  are protometrics.

By Fact 6, for any strict protometric  $p$ ,  $d(x, y) = p(x, y) + p(y, x) - p(x, x) - p(y, y)$  is a metric. In [2] it was shown that for the classes of  $\Sigma_m$ -proximities, this transformation is invertible. The  $\Sigma_m$ -proximities are strict protometrics  $\sigma$  such that for every  $x \in X$ ,  $\sigma(x, \cdot) = m$ , where  $\sigma(x, \cdot)$  is the value of an averaging linear functional applied to  $\sigma(x, y)$  as a function of  $y$ . Alternatively, this result can be derived in terms of difference protometrics (see below).

### 0-protometrics and difference protometrics

0-protometrics are the antipodes of strict protometrics (Definition 2). A *0-protometric* is a protometric  $p$  for which the pre-quadrangle inequality with  $z = y$  always holds as equality:  $p(x, y) + p(y, x) - p(x, x) - p(y, y) \equiv 0$ . The 0-protometrics form the largest linear space in the flat convex cone of protometrics on  $X$ . For a finite  $X$ , a basis of this space is given by all but one 0-protometrics  $q'_u(x, y) = 1_{x=u}$  and  $q''_u(x, y) = 1_{y=u}$  (since  $\sum_u q'_u \equiv \sum_u q''_u \equiv 1$ ). A basis of the space of symmetric 0-protometrics on  $X$  is given by all  $q'_u + q''_u$ .

A *difference protometric* (called *strong protometric* in [5]) is a protometric  $d$  such that  $d(x, x) \equiv 0$ . It satisfies  $d(x, y) + d(y, x) \geq 0$ . If, moreover,  $d(x, y) \geq 0$  for all  $x, y \in X$ , then  $d$  is a *quasi-semi-metric* [5, Section 1.1]. In this case, the binary relation  $x \preceq y \Leftrightarrow d(x, y) = 0$  is a preorder on  $X$ . Note that it is the *specialization preorder* in the induced topology.

Given a difference protometric  $d$  on  $X$  and any function  $f: X \rightarrow \mathbb{R}$ ,

$$p(x, y) = \frac{1}{2}(d(x, y) + f(x) + f(y)) \quad (1)$$

defines a protometric by virtue of Fact 4. It follows from (1) that for any  $x, y \in X$ ,

$$f(x) = p(x, x); \quad (2)$$

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y). \quad (3)$$

Thus, mappings (1) and (2)–(3) establish a bijection between all pairs  $(d, f)$ , where  $d$  is a difference protometric on  $X$ ,  $f$  being a function  $X \rightarrow \mathbb{R}$ , and some protometrics  $p$  on  $X$ . Since each protometric  $p$  generates a difference protometric  $d$  by means of (3), this bijection involves all protometrics  $p$  on  $X$ . The same mappings establish a bijection between the pairs  $(d, f)$ , where  $d$  is a semi-metric, and the symmetric protometrics on  $X$ . For example, a pair  $(d, f)$  with  $d$  being a semi-metric and  $f(x) = -d(x, x_0)$  corresponds to the protometric  $-(x, y)_{x_0}$ . Since  $f(x) + f(y)$  is a 0-protometric, (1) in the symmetric case with a finite  $X$  enables the representation

$$p = \frac{1}{2} \left( d + \sum_{u \in X} p(u, u)(q'_u + q''_u) \right).$$

The difference 0-protometrics (the elements of the intersection of difference protometrics and 0-protometrics) are exactly *potential differences*, i.e., functions of the form  $d(x, y) = h(x) - h(y)$ , where  $h: X \rightarrow \mathbb{R}$ .

Note that the protometrics of this paper are not related to the *proto-metrizable*, i.e., paracompact and having an orthobase, topological spaces.

## References

- [1] M. Catral, M. Neumann, J. Xu, Proximity in group inverses of M-matrices and inverses of diagonally dominant M-matrices, *Linear Algebra Appl.* 409 (2005) 32–50.
- [2] P.Yu. Chebotarev, E.V. Shamis, On a duality between metrics and  $\Sigma$ -proximities, *Autom. Remote Control* 59 (1998) 608–612. Erratum: 59 (1998) 1501.
- [3] P.Yu. Chebotarev, E.V. Shamis, On proximity measures for graph vertices, *Autom. Remote Control* 59 (1998) 1443–1459.
- [4] P. Chebotarev, The graph bottleneck identity, *Adv. in Appl. Math.* 47 (2011) 403–413.
- [5] M. Deza, E. Deza, *Encyclopedia of Distances*, Springer, Berlin–Heidelberg, 2009.
- [6] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Structures* 7 (1999) 71–83.
- [7] S.G. Matthews, Partial metric topology, Research Report 212, Dept. of Computer Science, University of Warwick, 1992.